

Boundedness of global solutions of a p -Laplacian evolution equation with a nonlinear gradient term

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Abstract

We investigate the boundedness and large time behavior of solutions of the Cauchy-Dirichlet problem for the one-dimensional degenerate parabolic equation with gradient nonlinearity:

$$u_t = (|u_x|^{p-2}u_x)_x + |u_x|^q \quad \text{in } (0, \infty) \times (0, 1), \quad q > p > 2.$$

We prove that: either u_x blows up in finite time, or u is global and converges in $W^{1,\infty}$ norm to the unique steady state. This in particular eliminates the possibility of global solutions with unbounded gradient. For that purpose a Lyapunov functional is constructed by the approach of Zelenyak.

1 Introduction and main results

In this paper we are interested in the asymptotic behavior of global solutions to the following one-dimensional degenerate diffusive Hamilton-Jacobi equation

$$\begin{cases} u_t - (|u_x|^{p-2}u_x)_x = |u_x|^q, & 0 < x < 1, t > 0, \\ u(t, 0) = 0, \quad u(t, 1) = M \geq 0, & t > 0, \\ u(0, x) = u_0(x), & 0 < x < 1, \end{cases} \quad (1.1)$$

with $q > p > 2$, $M \geq 0$ and suitably regular initial data u_0 .

Problem (1.1) models a variety of physical phenomena which arise for example in the study of surface growth where a stochastic version of it is known as the Kardar-Parisi-Zhang equation ($p = 2, q = 2$). It has also a mathematical interest through the viscosity approximation of Hamilton-Jacobi type equations from control theory.

Solutions of (1.1) exhibit a rich variety of qualitative behaviors, according to the values of $p \geq 2$ and $q \in (0, \infty)$.

If $q \leq p$, it is known that all solutions are global and bounded in $W^{1,\infty}$ norm [10]. For $q \in [p - 1, p]$ it was proved in [13] that nonnegative viscosity solutions of (1.1) with homogeneous Dirichlet boundary condition decay to 0 and the rate of convergence was also obtained, see also [6] for the semilinear case. Concerning the large time behavior of global weak solutions to (1.1) with homogeneous boundary conditions and $q \in (0, p - 1)$,

it has been shown that there exists a one parameter family of nonnegative steady states, and any solution converges uniformly to one of these stationary solutions (cf. [18, 5, 11]).

For $q > p \geq 2$, the situation is quite different. It is known that for any $M \geq 0$ and suitably large u_0 , there exist solutions of (1.1) for which the L^∞ norm of the gradient blows up in finite time (the L^∞ norm of the solution remaining bounded) [15, 4], while there exist global and decaying solutions for u_0 sufficiently small [17]. In view of a classification of all solutions of (1.1), it is then a natural question to ask whether or not C^1 -unbounded global solutions may exist. The question of the boundedness of global solutions of (1.1) was initiated for the semilinear case $p = 2$ in [2] and further investigated in [16, 17]. Denoting $M_c := (q-1)^{\frac{q-2}{q-1}}/(q-2)$, the result of [2] says that if $0 \leq M < M_c$, then any global solution of (1.1) is bounded in C^1 norm for $t \geq 0$, that is,

$$\sup_{t \geq 0} \|u_x(t, \cdot)\|_\infty < \infty. \quad (1.2)$$

On the other hand, it is known from [16] that some unbounded global solutions do exist if $M = M_c$ and $u_0 \leq U(x)$ where $U(x) := M_c x^{(q-2)/(q-1)}$ is the unique singular steady state. Moreover the precise exponential rates of the gradient blow-up in infinite time was obtained.

Motivated by the results of the papers [2, 16], we modify the method used by Arrieta, Rodriguez-Bernal and Souplet and extend their results on the classification of large time behavior of global solutions to the degenerate parabolic equation case $p > 2$.

From now on, we assume that $q > p > 2$. By a solution of (1.1), we mean a weak solution (see Section 2 below for a precise definition and well-posedness results). We recall that weak solutions of (1.1) satisfy a comparison principle, hence in particular

$$\min_{[0,1]} u_0 \leq u(t, x) \leq \max_{[0,1]} u_0, \quad 0 \leq t < T_{max}, \quad 0 \leq x \leq 1. \quad (1.3)$$

Our main result is then the following:

Theorem 1.1. *Assume that $q > p > 2$ and u_0 in $W^{1,\infty}(0,1)$, $u_0(0) = 0, u_0(1) = M$. Set $M_b = \frac{q-p+1}{q-p} \left(\frac{q-p+1}{p-1} \right)^{\frac{1}{p-1-q}}$.*

(i) *If $0 < M < M_b$ or $M = 0$ and $u_0 \in C^2([0,1])$, then all global weak solution of (1.1) are bounded in $W^{1,\infty}$ norm. Moreover they converge in $W^{1,\infty}([0,1])$ to the unique steady state.*

(ii) *If $M > M_b$, then all weak solutions of (1.1) exhibit gradient blow-up in finite time.*

The proof of Theorem 1.1 proceeds by contradiction. It relies on the analysis of steady states and the existence of a Lyapunov functional which enjoys nice properties on any global trajectory of (1.1), even if it were unbounded in $W^{1,\infty}$ norm. The construction of such a nice Lyapunov functional which is handled through the Zelenyak technique, together with the fact that the singularities may only take place near the boundary, allow us to prove the following convergence result: any global solution, even unbounded in $W^{1,\infty}$ must converge in $C([0,1])$ to a stationary solution W of (1.1) with $W(0) = 0$, $W(1) = M$ (see Proposition 3.3). On the other hand, if u were unbounded, then our

gradient estimates would imply that $W_x(0) = +\infty$ or $W_x(1) = -\infty$. But such a W does not exist if $M \neq M_b$, leading to a contradiction.

Although the scheme of proof follows that in [2] for $p = 2$, we have to face a number of additional technical difficulties, caused by the lack of regularity of solutions. In particular, we have to work at the level of regularized problems, including for the construction of Lyapunov-Zelenyak functional. This, in turn requires good convergence properties and estimates of regularized solutions. For this, we heavily rely on results from our previous work [4] (which concerned the higher dimensional problem as well).

Remark 1.1. (a) *For the case $M = 0$, the restrictive assumption $u_0 \in C^2([0, 1])$ is probably just technical.*

(b) *For the critical case $M = M_b$, all solutions must blow up in either finite or infinite time. The existence of global solution which are unbounded in $W^{1,\infty}$ norm (that is infinite time gradient blow-up) should occur for some suitable initial data as it is the case for the corresponding semilinear equation, but this still is an open problem. Moreover, we know from Proposition 3.3 that, even in this case the solutions will converge in $C([0, 1])$ to the unique singular steady state.*

(c) *Since the technique of Zelenyak to obtain a Lyapunov functional is restricted to the one-dimensional setting, the large time behavior and the boundedness of global solution in $W^{1,\infty}$ norm are still open problems in higher dimension.*

Let us mention some results concerning related equations possessing solutions with unbounded gradient. When the nonlinearity is replaced with an exponential one and $p = 2$, results on boundedness and existence of infinite time gradient blow-up solutions are obtained in [20, 21]. A phenomenon of infinite time gradient blow-up has been observed for quasilinear equations involving mean curvature type operators [7]. For results on interior gradient blow-up we refer the reader to [1, 3]. Finally for other results concerning existence, asymptotic behavior of global solutions for the corresponding Cauchy problem and a viscosity solution approach see [8, 12, 14] and references therein.

The rest of the paper is organized as follows. Section 2 contains some useful preliminary material, including smoothing properties of solutions and estimate of the derivative u_x . In section 3, we employ the technique of Zelenyak [19], along with a trick used in [2], to construct an approximate Lyapunov functional for weak solutions to (1.1). Section 4 is devoted to the proof of Theorem 1.1.

2 Preliminary estimates and steady states

2.1 Gradient estimates

For $u_0 \in W^{1,\infty}((0, 1))$, $u_0(0) = 0, u_0(1) = M$, by a (weak) solution of (1.1) on $[0, T]$, we mean a function $u \in C([0, T] \times [0, 1]) \cap L^q(0, T; W^{1,q}(0, 1))$ such that

$$u_t \in L^2(0, T; L^2((0, 1))), \quad u(x, 0) = u_0(x), \quad u(t, 1) = 0, u(t, 0) = M$$

and

$$\int_0^T \int_0^1 u_t \psi + |u_x|^{p-2} u_x \cdot \psi_x dx dt = \int_0^T \int_0^1 |u_x|^q \psi dx dt, \quad (2.1)$$

holds for all $\psi \in C^0([0, T] \times [0, 1]) \cap L^p((0, T); W_0^{1,p}((0, 1)))$ satisfying $\psi(t, 0) = \psi(t, 1) = 0$ on $[0, T]$.

It is known (see e.g., [4]) that there exists $T_{max} = T_{max}(u_0) \in (0, \infty]$ such that for each $T \in (0, T_{max})$, (1.1) admits a unique solution u such that $u \in L^\infty((0, T); W^{1,\infty}((0, 1)))$. Moreover we know that u is a C^1 -function w.r.t. the space variable in $(0, T) \times (0, 1)$ and that its derivative u_x is locally Hölder continuous.

In order to describe the asymptotic behavior, we need to collect some preliminary estimates. We will need the following theorem which gives a useful regularizing property for local solutions of (1.1) (see [4]).

Theorem 2.1. *Assume that $q > p - 1$ and let $u \in L^\infty([0, T_{max}); W^{1,\infty}((0, 1)))$ be the maximal weak solution of problem (1.1). Then*

$$u_t \leq \frac{1}{p-2} \frac{\|u_0\|_\infty}{t} \quad \text{in } \mathcal{D}'(\Omega) \quad \text{for a.e. } t \in (0, T_{max}). \quad (2.2)$$

Thanks to the upper bound of u_t , we derive the following two lemmas giving lower and upper bounds on u_x , showing that u_x remains bounded away from the boundary.

Lemma 2.2. *Let u be a maximal weak solution of (1.1). For all $t_0 \in (0, T_{max})$, there exists $C_1 > 0$ such that for all $t \in [t_0, T_{max})$ and $0 < x < 1$,*

$$u_x(t, x) \leq \left(\left(\frac{q-p+1}{p-1} x \right)^{\frac{1-p}{q-p+1}} + C_1 x \right)^{1/p-1}, \quad (2.3)$$

$$u_x(t, 1-x) \geq - \left(\left(\frac{q-p+1}{p-1} x \right)^{\frac{1-p}{q-p+1}} + C_1 x \right)^{1/p-1}. \quad (2.4)$$

Proof. Fix $t \in [t_0, T_{max})$ and let $y(x) = (|u_x|^{p-2} u_x(t, x) - C_1 x)^+$, where $C_1 = \frac{\|u_0\|_\infty}{(p-2)t_0}$.

On any interval (a, b) with $0 < a < b < 1$ where $y > 0$, the function y satisfies in the classical sense $y' + y^{\frac{q}{p-1}} \leq 0$. Indeed, for each $x \in (a, b)$, we have $|u_x|^{p-2} u_x > C_1 x > C_1 a > 0$ and the function u is smooth at such points since the equation is uniformly parabolic [10]. Using lemma 2.1, we get that $y' + y^{\frac{q}{p-1}} \leq ((|u_x|^{p-2} u_x)_x - C_1) + |u_x|^q \leq 0$.

Integrating this inequality, it follows that $y(x) \leq \left(\frac{q-p+1}{p-1} x \right)^{\frac{1-p}{q-p+1}}$ on (a, b) and $y(a) > 0$.

If $y \not\equiv 0$, then we can find $c = c(t) \in (0, 1]$ such that $y > 0$ in $(0, c)$ and $y = 0$ in $[c, 1)$.

Therefore we get $y(x) \leq \left(\frac{q-p+1}{p-1} x \right)^{\frac{1-p}{q-p+1}}$ on $(0, 1)$ and (2.3) is readily deduced. In the same manner, considering $y(x) = (-|u_x|^{p-2} u_x(t, 1-x) - C_1 x)^+$, we get (2.4).

Remark 2.1. *Similar gradient estimates in any space dimension are already obtained in [4] using a more technical Bernstein type argument.*

2.2 Steady states

It is a well-known fact that the large-time behavior of evolution equations is closely connected to the existence and properties of the stationary states. In this part we are looking for nonnegative stationary solutions V of (1.1), that is weak solution of

$$\begin{cases} (|V_x|^{p-2}V_x)_x + |V_x|^q = 0, & x \in (0, 1), \\ V(0) = 0, & V(1) = M \geq 0. \end{cases} \quad (2.5)$$

More precisely, $V \in C([0, 1]) \cap C^1(0, 1)$ is a weak solution of (2.5) if $V(0) = 0, V(1) = M$ and V satisfies

$$\int_0^1 (|V_x|^{p-2}V_x) \phi_x dx - |V_x|^q \phi = 0 \quad \text{for any } \phi \in W_0^{1,p}(0, 1). \quad (2.6)$$

It is not difficult to show that any weak solution in the above sense is actually a classical C^2 solution in $(0, 1)$ (for any $x_0 \in (0, 1)$, consider separately the cases $V_x(x_0) \neq 0$ and $V_x(x_0) = 0$). For small values of $M \geq 0$, problem (2.5) admits a unique weak solution $V_M = V_M(x) \in C^2([0, 1])$. Namely, this happens for $0 \leq M < M_b$, where M_b is the critical value,

$$M_b = \frac{q-p+1}{q-p} \left(\frac{q-p+1}{p-1} \right)^{-1/(q-p+1)}.$$

On the other hand, there is no steady state if $M > M_b$. In the critical case $M = M_b$, there still exists a steady state $V_{M_b} = U$, given by the explicit formula $U(x) = M_b x^{\frac{q-p}{q-p+1}}$. U belongs to $C([0, 1]) \cap C^2((0, 1])$, but it is singular in the sense that it has infinite derivative on the left-hand boundary, $U_x(0) = \infty$.

Proposition 2.3. *For $0 \leq M \leq M_b$, let $V \in C([0, 1]) \cap C^1(0, 1)$ be a non-negative weak solution to (2.5). Then either $V \equiv 0$ if $M = 0$, or there exists $k = k(M) \in [0, \infty)$ for $M > 0$ such that*

$$V = V_k := M_b \left[(x+k)^{\frac{q-p}{q-p+1}} - k^{\frac{q-p}{q-p+1}} \right].$$

3 Lyapunov functional and convergence to steady states

Since (1.1) is a degenerate problem, we do not have sufficient regularity properties of the trajectories to construct a good smooth Lyapunov functional (which exists for uniformly parabolic equations). Hence we first consider a regularized problem, then the main estimate which plays a key role in the proof of the convergence to steady states will be proved by passing to the limit $\varepsilon \rightarrow 0$ in the regularizing parameter.

Let $\varepsilon \in (0, 1/2)$. We consider the following approximate problems:

$$\begin{cases} (u_\varepsilon)_t = (p-1) (|(u_\varepsilon)_x|^2 + \varepsilon^2)^{\frac{p-2}{2}} (u_\varepsilon)_{xx} + (|(u_\varepsilon)_x|^2 + \varepsilon^2)^{\frac{q}{2}} & t > 0, \quad x \in (0, 1), \\ u_\varepsilon(t, 0) = 0, \quad u_\varepsilon(t, 1) = M, & t > 0, \\ u_\varepsilon(0, x) = u_0(x), & x \in (0, 1). \end{cases} \quad (3.1)$$

First, let us note that due to $q > p > 2$, we have for ε small enough $(p-1)\varepsilon^p \cosh(\varepsilon x)^{p-1} \geq \varepsilon^q \cosh(\varepsilon x)^q$ (it suffices to take $0 < \varepsilon < \cosh(1)^{\frac{p-1-q}{q-p}}$). Hence $\|u_0\|_{L^\infty} + M + 2 - \cosh(\varepsilon x)$

is a supersolution for problem (3.1). Therefore there exists $K > 0$ depending only on $\|u_0\|_{L^\infty}$ such that, for all $\varepsilon \in (0, 1/2)$, $\|u_\varepsilon(t, x)\|_{L^\infty} \leq K$.

Next we collect some useful properties of the sequence $\{u_\varepsilon\}$ which we will use later on.

Proposition 3.1. *Let $u \in L^\infty((0, T); W^{1,\infty}((0, 1)))$ for any $T > 0$ be a global solution of (1.1) and let u_ε a classical solution of (3.1) on $(0, T(u_\varepsilon))$, then for ε small we have*

- a) $T(u_\varepsilon) > T$, $u_\varepsilon \rightarrow u$ in $C([0, T] \times [0, 1])$ and $(u_\varepsilon)_x \rightarrow u_x$ in $C_{loc}((0, T) \times (0, 1))$.
- b) $|(u_\varepsilon)_x(t, x)|_{L^\infty} \leq C = C(\|u_0\|_{W^{1,\infty}})$ on $(0, T) \times (0, 1)$ and

$$\int_0^T \int_0^1 (u_\varepsilon)_t^2 dx dt \leq (\|u_0\|_{W^{1,\infty}}^p + 1 + TC^{2q}).$$

Proof. We know from [4] that there exist a subsequence $\{u_{\varepsilon_n}\}$ of $\{u_\varepsilon\}$ and a small time $\tau_1 = \tau_1(\|u_0\|_{W^{1,\infty}}) > 0$ such that u_{ε_n} converges in $C([0, \tau_1] \times [0, 1]) \cap C_{loc}^{0,1}((0, \tau_1) \times (0, 1))$ to a solution \tilde{u} of (1.1). The uniqueness of the solution of (1.1) implies that $\tilde{u} = u$ and that the whole sequence converges to u . Let $\tilde{T} := \sup\{s > 0 \text{ such that } \exists \{u_\varepsilon\} \text{ solutions of (3.1), } u_\varepsilon \rightarrow u \text{ in } C([0, s] \times [0, 1]) \cap C_{loc}^{0,1}((0, s) \times (0, 1))\}$. Assume that $\tilde{T} < T$. For $\eta > 0$ small, we have

$$\|u_\varepsilon(\tilde{T} - \eta)\|_{W^{1,\infty}} \leq C \|u(\tilde{T} - \eta)\|_{W^{1,\infty}}.$$

Thus we can find $\tau > 0$ independent of ε and η such that the problem

$$\begin{cases} (u_\varepsilon^\eta)_t = (p-1) (|(u_\varepsilon^\eta)_x|^2 + \varepsilon^2)^{\frac{p-2}{2}} (u_\varepsilon^\eta)_{xx} + (|(u_\varepsilon^\eta)_x|^2 + \varepsilon^2)^{\frac{q}{2}} & t > 0, \quad x \in (0, 1), \\ u_\varepsilon^\eta(t, 0) = 0, \quad u_\varepsilon^\eta(t, 1) = M, & t > 0, \\ u_\varepsilon^\eta(0, x) = u_\varepsilon(\tilde{T} - \eta, x), & x \in (0, 1). \end{cases} \quad (3.2)$$

admits a unique classical solution u_ε^η on $[0, \tau]$. We can extend the solution u_ε of (3.1) on $[0, \tilde{T} - \eta + \tau]$ by setting $u_\varepsilon(t, x) = \begin{cases} u_\varepsilon(t, x) & \text{for } x \in [0, \tilde{T} - \eta], \\ u_\varepsilon^\eta(t, x) & \text{for } x \in [\tilde{T} - \eta, \tilde{T} - \eta + \tau] \end{cases}$.

For η small enough, we have $\tilde{T} - \eta + \tau > \tilde{T}$, which contradicts the definition of \tilde{T} . Hence $\tilde{T} \geq T$.

The second assertion follows from the estimates given in [4, Inequalities 2.16 and 2.19].

Now we construct a Lyapunov functional for (3.1) with the help of the technique developed by Zelenyak [19]. Let $D_K = [-K, K] \times \mathbb{R}$. We look for a pair of functions $\Phi_\varepsilon \in C^1(D_K; \mathbb{R})$ and $\Psi_\varepsilon \in C(D_K; (0, \infty))$ with the following property: For any solution u_ε of (3.1) with $|u_\varepsilon| \leq K$, defining

$$\mathcal{L}_\varepsilon(u_\varepsilon(t)) = \int_0^1 \Phi_\varepsilon(u_\varepsilon(t, x), (u_\varepsilon)_x(t, x)) dx,$$

it holds

$$\frac{d}{dt} \mathcal{L}_\varepsilon(u_\varepsilon(t)) = - \int_0^1 \Psi_\varepsilon(u_\varepsilon(t, x), (u_\varepsilon)_x(t, x)) (u_\varepsilon)_t^2(t, x) dx.$$

Since $(u_\varepsilon)_t(t, 0) = (u_\varepsilon)_t(t, 1) = 0$, we have

$$\begin{aligned} \frac{d}{dt} \int_0^1 \Phi_\varepsilon(u_\varepsilon, (u_\varepsilon)_x) dx &= \int_0^1 (u_\varepsilon)_t \cdot (\Phi_\varepsilon)_u(u_\varepsilon, (u_\varepsilon)_x) + (u_\varepsilon)_{xt} \cdot (\Phi_\varepsilon)_v(u_\varepsilon, (u_\varepsilon)_x) dx \\ &= \int_0^1 (u_\varepsilon)_t \left[(\Phi_\varepsilon)_u(u_\varepsilon, (u_\varepsilon)_x) - (u_\varepsilon)_x \cdot (\Phi_\varepsilon)_{uv}(u_\varepsilon, (u_\varepsilon)_x) - (u_\varepsilon)_{xx} \cdot (\Phi_\varepsilon)_{vv}(u_\varepsilon, (u_\varepsilon)_x) \right] dx. \end{aligned}$$

So it is natural to require that

$$\begin{aligned} &(\Phi_\varepsilon)_u(u_\varepsilon, (u_\varepsilon)_x) - (u_\varepsilon)_x \cdot (\Phi_\varepsilon)_{uv}(u_\varepsilon, (u_\varepsilon)_x) - (u_\varepsilon)_{xx} \cdot (\Phi_\varepsilon)_{vv}(u_\varepsilon, (u_\varepsilon)_x) \\ &= -\Psi_\varepsilon(u_\varepsilon, (u_\varepsilon)_x) \cdot (u_\varepsilon)_t \\ &= -\Psi_\varepsilon(u_\varepsilon, (u_\varepsilon)_x) \left[(p-1) (|(u_\varepsilon)_x|^2 + \varepsilon^2)^{\frac{p-2}{2}} (u_\varepsilon)_{xx} + (|(u_\varepsilon)_x|^2 + \varepsilon^2)^{\frac{q}{2}} \right] \end{aligned}$$

A sufficient condition is

$$(\Phi_\varepsilon)_{vv}(u, v) = (p-1) \Psi_\varepsilon(u, v) (v^2 + \varepsilon^2)^{\frac{p-2}{2}}, \quad (3.3)$$

$$(\Phi_\varepsilon)_u(u, v) - v(\Phi_\varepsilon)_{uv}(u, v) = -\Psi_\varepsilon(u, v) (v^2 + \varepsilon^2)^{\frac{q}{2}}, \quad (3.4)$$

that is Φ_ε satisfies the differential equation:

$$(\Phi_\varepsilon)_u(u, v) - v(\Phi_\varepsilon)_{uv}(u, v) + \frac{(v^2 + \varepsilon^2)^{\frac{q-p+2}{2}}}{p-1} (\Phi_\varepsilon)_{vv}(u, v) = 0. \quad (3.5)$$

We follow the method used in [2] to find such nice functions. For a given function $\rho_\varepsilon(u, v)$, let us denote

$$H_\varepsilon = (\rho_\varepsilon)_u + \frac{(v^2 + \varepsilon^2)^{\frac{q-p+2}{2}}}{p-1} (\rho_\varepsilon)_{vv} - v(\rho_\varepsilon)_{uv}.$$

Here we assume that $\rho_\varepsilon, (\rho_\varepsilon)_u, (\rho_\varepsilon)_v, (\rho_\varepsilon)_{uv}$ are continuous and C^1 in v in D_K , and that $(\rho_\varepsilon)_{vv}$ is continuous in D_K and, except perhaps at $v = 0$, C^1 in v .

We want to have $(H_\varepsilon)_v = 0$, so that $H_\varepsilon(u, v) = H_\varepsilon(u, 0) = H_\varepsilon(u)$. We compute

$$(H_\varepsilon)_v = \frac{(v^2 + \varepsilon^2)^{\frac{q-p+2}{2}}}{p-1} (\rho_\varepsilon)_{vvv} + \left(\frac{q-p+2}{p-1} \right) v (v^2 + \varepsilon^2)^{\frac{q-p}{2}} (\rho_\varepsilon)_{vv} - v(\rho_\varepsilon)_{uvv}.$$

For this, it suffices that $f_\varepsilon = (\rho_\varepsilon)_{vv}$ satisfies the following conditions:

$$\begin{cases} (f_\varepsilon)_u - \left(\frac{q-p+2}{p-1} \right) (v^2 + \varepsilon^2)^{\frac{q-p}{2}} f_\varepsilon - \frac{(v^2 + \varepsilon^2)^{\frac{q-p+2}{2}}}{(p-1)v} (f_\varepsilon)_v = 0 & |u| \leq K, \quad v \neq 0, \\ (f_\varepsilon)_v(u, 0) = 0. \end{cases} \quad (3.6)$$

Now, the equation (3.6) can be solved by the method of characteristics. For each $K > 0$, one finds that the function defined by

$$f_\varepsilon(u, v) = \left[1 + \left(\frac{q-p}{p-1} \right) (v^2 + \varepsilon^2)^{\frac{q-p}{2}} (K+1-u) \right]^{-\frac{q-p+2}{q-p}} > 0$$

is a solution of (3.6) on $[-K, K] \times \mathbb{R}$. Define ρ_ε by

$$\rho_\varepsilon(u, v) = \int_0^v \int_0^z f_\varepsilon(u, s) ds dz \geq 0,$$

and let then

$$\Phi_\varepsilon(u, v) = \rho_\varepsilon(u, v) - \int_0^u H_\varepsilon(s, 0) ds + C. \quad (3.7)$$

We added a constant $C > 0$ to ensure that $\Phi_\varepsilon \geq 0$. This constant C does not depend on ε . In fact, given that $\varepsilon \leq 1/2$, $2 < p$ and $0 \leq (\rho_\varepsilon)_{vv} \leq 1$, we get

$$1 = (\rho_\varepsilon)_u(s, 0) + 1 \geq H_\varepsilon(s, 0) \geq (\rho_\varepsilon)_u(s, 0) = 0,$$

and consequently

$$- \int_0^u H_\varepsilon(s, 0) ds \geq \rho_\varepsilon(0, 0) - \rho_\varepsilon(u, 0) - u = -u \quad \text{for } u \geq 0.$$

But since $|u| \leq K$, it suffices to take $C = K + 1 > K$. Using the definition of H_ε and the fact that $H_\varepsilon(u, v) = H_\varepsilon(u, 0)$, we see that:

$$(\Phi_\varepsilon)_u - v(\Phi_\varepsilon)_{uv}(u, v) + \frac{(v^2 + \varepsilon^2)^{\frac{q-p+2}{2}}}{p-1} (\Phi_\varepsilon)_{vv}(u, v) = 0,$$

i.e. Φ_ε satisfies (3.5), hence (3.3)-(3.4) with

$$\Psi_\varepsilon(u, v) = \frac{(v^2 + \varepsilon^2)^{\frac{2-p}{2}} (\rho_\varepsilon)_{vv}}{p-1} > 0. \quad (3.8)$$

It follows that

$$\frac{d}{dt} \mathcal{L}_\varepsilon(u_\varepsilon(t)) = - \int_0^1 \frac{((u_\varepsilon)_x^2 + \varepsilon^2)^{\frac{2-p}{2}} (\rho_\varepsilon)_{vv}}{p-1} (u_\varepsilon)_t^2 dx = - \int_0^1 \Psi_\varepsilon(u_\varepsilon, (u_\varepsilon)_t) (u_\varepsilon)_x^2 dx.$$

As a direct consequence of the existence of the approximate Lyapunov functional, we have the following estimate. Set

$$\mathcal{A}(u, v) = \frac{(v^2 + 1)^{\frac{2-p}{2}}}{p-1} \left[1 + \frac{(q-p)}{(p-1)} (v^2 + 1)^{\frac{q-p}{2}} (K + 1 - u) \right]^{-\frac{q-p+2}{q-p}}.$$

Proposition 3.2. *Let $q > p > 2$ and u_ε denote the classical solution of (3.1). Then for any $T \in (0, T_{\max})$, we have*

$$\int_0^T \int_0^1 \mathcal{A}(u_\varepsilon, (u_\varepsilon)_x) \cdot (u_\varepsilon)_t^2(t, x) dx dt \leq C(\|u_0\|_{W^{1,\infty}}). \quad (3.9)$$

Proof. First, due to $\varepsilon \in (0, 1/2)$ and $q > p > 2$, we remark that $\Psi_\varepsilon(u, v) \geq \mathcal{A}(u, v)$. Next, since $f_\varepsilon(u, v) \leq 1$ and $|\int_0^u H_\varepsilon(s) ds| \leq |u|$, we get that $\Phi_\varepsilon(u, v) \leq v^2 + |u| + K + 1$. Using that $\Phi_\varepsilon(u, v) \geq 0$, we get

$$\begin{aligned} \int_0^T \int_0^1 \mathcal{A}(u_\varepsilon, (u_\varepsilon)_x) \cdot (u_\varepsilon)_t^2(t, x) dx dt &\leq \int_0^T \int_0^1 \Psi_\varepsilon(u_\varepsilon, (u_\varepsilon)_x) \cdot (u_\varepsilon)_t^2(x, t) dx dt \\ &= \mathcal{L}_\varepsilon(u_0) - \mathcal{L}_\varepsilon(u_\varepsilon(T)) \\ &\leq C(\|u_0\|_{W^{1,\infty}}). \end{aligned}$$

3.1 Convergence to steady states

Proposition 3.3. *Let u be a global weak solution of (1.1). Then $u(t)$ converges in $C([0, 1])$ to a steady state of (1.1) as $t \rightarrow \infty$. Moreover the convergence also holds in $C^1([\delta, 1 - \delta])$ for all $\delta > 0$.*

Proof. Assume that u is a global weak solution of (1.1). Fix a sequence $(t_k)_{k \in \mathbb{N}}$, $1 \leq t_k \rightarrow \infty$ and set $w_k(t, x) = u(t + t_k, x)$. By a comparison principle, we know that

$$|u| \leq |u_0|_\infty \quad \text{in } [1, \infty) \times [0, 1], \quad (3.10)$$

Using lemma 2.2 we have

$$|u_x| \leq C(\delta), \quad \text{in } [1, \infty) \times [\delta/2, 1 - \delta/2]. \quad (3.11)$$

Thus applying a result of DiBenedetto-Friedman [9], we have that $\{w_k\}$ and $\{(w_k)_x\}$ are Hölder continuous in $[\delta, T - \delta] \times [\delta, 1 - \delta]$ with a Hölder norm independent of k . It follows that $\{w_k\}$ and $\{(w_k)_x\}$ are relatively compact in $C([\delta, T - \delta] \times [\delta, 1 - \delta])$ for any $\delta, T > 0$. Thus, by the Arzelà-Ascoli theorem there exist a subsequence $(t_{k_l})_{l \in \mathbb{N}}$ of (t_k) and a function $W \in C((0, \infty) \times (0, 1))$, $W_x \in C((0, \infty) \times (0, 1))$ such that for any $\delta, T > 0$

$$w_{k_l} \rightarrow W \quad \text{strongly in } C^0([\delta, T - \delta] \times [\delta, 1 - \delta]) \quad \text{as } l \rightarrow \infty. \quad (3.12)$$

$$(w_{k_l})_x \rightarrow W_x \quad \text{strongly in } C^0([\delta, T - \delta] \times [\delta, 1 - \delta]) \quad \text{as } l \rightarrow \infty. \quad (3.13)$$

and W satisfies

$$W_t - (|W_x|^{p-2}W_x)_x = |W_x|^q, \quad t > 0, x \in (0, 1).$$

Further, using lemma 2.2 and $q > p$, we get

$$|(w_k)_x|_{L^\infty(1, \infty; L^1(0, 1))} \leq C. \quad (3.14)$$

Combining (3.10) with (3.14), we get that, for each fixed $t > 0$, $u(x, t + t_k)$ is relatively compact in $C([0, 1])$. Consequently for any $t > 0$, $W(t, \cdot)$ can be extended to a continuous function on $[0, 1]$ satisfying

$$W(t, 0) = 0 \quad W(t, 1) = M.$$

On the other hand, since $0 < \mathcal{A}(u, v) \leq 1$ and $(u_\varepsilon)_t$ is bounded in $L^2((0, T) \times (0, 1))$ [4, Inequality 2.19], it follows that $(u_\varepsilon)_t$ is bounded in $L^2((1, T) \times (\delta, 1 - \delta); \mathcal{A}(u, u_x) dx dt)$. Since $L^2((1, T) \times (\delta, 1 - \delta); \mathcal{A}(u, u_x) dx dt)$ is a reflexive space, we get that (up to a subsequence) $(u_\varepsilon)_t \rightharpoonup u_t$ in $L^2((1, T) \times (\delta, 1 - \delta); \mathcal{A}(u, u_x) dx dt)$, hence

$$\int_1^T \int_\delta^{1-\delta} \mathcal{A}(u, u_x)(u_t)^2 dx dt \leq \liminf_{\varepsilon \rightarrow 0} \int_1^T \int_\delta^{1-\delta} \mathcal{A}(u, u_x)(u_\varepsilon)_t^2 dx dt.$$

Using Proposition 3.1, we have

$$\begin{aligned} \int_1^T \int_\delta^{1-\delta} |\mathcal{A}(u, u_x) - \mathcal{A}(u_\varepsilon, (u_\varepsilon)_x)| (u_\varepsilon)_t^2 dx dt &\leq \\ (\|u_0\|_{W^{1, \infty}}^p + 1 + TC^{2q}) \sup_{[1, T] \times [\delta, 1-\delta]} |\mathcal{A}(u, u_x) - \mathcal{A}(u_\varepsilon, (u_\varepsilon)_x)|, \end{aligned} \quad (3.15)$$

hence

$$\liminf_{\varepsilon \rightarrow 0} \int_1^T \int_{\delta}^{1-\delta} |\mathcal{A}(u, u_x) - \mathcal{A}(u_{\varepsilon}, (u_{\varepsilon})_x)| (u_{\varepsilon})_t^2 dx dt = 0.$$

Next, using (3.15) and (3.9), we get,

$$\begin{aligned} \int_1^T \int_{\delta}^{1-\delta} \mathcal{A}(u, u_x) (u_{\varepsilon})_t^2 dx dt &\leq \int_0^T \int_{\delta}^{1-\delta} \mathcal{A}(u_{\varepsilon}, (u_{\varepsilon})_x) (u_{\varepsilon})_t^2 dx dt \\ &\quad + \int_1^T \int_{\delta}^{1-\delta} |\mathcal{A}(u, u_x) - \mathcal{A}(u_{\varepsilon}, (u_{\varepsilon})_x)| (u_{\varepsilon})_t^2 dx dt \\ &\leq (\|u_0\|_{W^{1,\infty}}^p + 1 + TC^{2q}) \sup_{[1,T] \times [\delta, 1-\delta]} |\mathcal{A}(u, u_x) - \mathcal{A}(u_{\varepsilon}, (u_{\varepsilon})_x)| \\ &\quad + C(\|u_0\|_{W^{1,\infty}}). \end{aligned}$$

Thus

$$\int_1^T \int_{\delta}^{1-\delta} \mathcal{A}(u, u_x) (u_t)^2 dx dt \leq C(\|u_0\|_{W^{1,\infty}}),$$

and it follows that

$$\theta(\|u_0\|_{L^{\infty}}, C(\delta)) \int_1^T \int_{\delta}^{1-\delta} (u_t)^2 dx dt \leq C(\|u_0\|_{W^{1,\infty}}), \quad (3.16)$$

where $\theta(K, R) := \inf \{ \mathcal{A}(u, v); \quad |u| \leq K, \quad |v| \leq R \} > 0$. This implies that

$$\int_0^{\infty} \int_{\delta}^{1-\delta} (w_{k_l})_t^2(t, x) dx dt \rightarrow 0, \quad \text{as } l \rightarrow \infty.$$

Since $(w_{k_l})_t \rightarrow W_t$ in $\mathcal{D}'((0, \infty) \times (0, 1))$ and $\delta \in (0, 1)$ is arbitrary, it follows that $W_t \equiv 0$. Thus W is a steady state of (1.1). Given that the sequence $t_{k_l} \rightarrow \infty$ is arbitrary and the steady states (for given M) are unique, it follows that the whole solution $u(t)$ converges to W .

4 Proof of theorem 1.1

First of all we shall derive a lower bound on the time derivative of global weak solutions.

Lemma 4.1. *Let u be a global solution of (1.1). If $M > 0$ or $M = 0$ and $u_0 \in C^2([0, 1])$, then there exist $t_1 > 0$, $C_3 > 0$ such that*

$$|u_t| \leq C_3, \quad \text{a.e. on } (t_1, +\infty) \times (0, 1). \quad (4.1)$$

Moreover, if $M > 0$, then u becomes a classical solution for $t > t_1$.

Proof. We will treat differently the two cases $M = 0$ and $M > 0$.

For $M > 0$, using Proposition 3.3 and our assumption that u is global, one can show that u_x becomes positive in $[0, 1]$ for large t . Indeed, we can see that there exists $\eta > 0$ such that $W_x \geq 2\eta > 0$ in $[0, 1]$. Proceeding as in the proof of lemma 2.2, we can show that $y := (|u_x|^{p-2}u_x - C_1x)_+$ is nonincreasing with respect to x . Fix $a \in (0, 1/2)$ small

enough such that $\eta^{p-1} - 2C_1a > 0$. Since $u(t, \cdot) \rightarrow W$ in $C([0, 1]) \cap C_{\text{loc}}^1(0, 1)$, there exists $\hat{T} > 0$ such that

$$u_x(t, x) > W_x(x) - \eta \geq \eta \quad \text{for all } x \in [a, 1/2], t > \hat{T}. \quad (4.2)$$

But since y is nonincreasing, we have for all $x \in (0, a]$, $t > \hat{T}$

$$|u_x|^{p-2}u_x(t, x) \geq y(t, x) \geq y(t, a) \geq \eta^{p-1} - C_1a \geq C_1a. \quad (4.3)$$

Combining (4.2)-(4.3) and then arguing by symmetry that the same result holds true for the interval $[1/2, 1]$, we arrive at

$$u_x(t, x) \geq \tilde{\eta} > 0, \quad t > \hat{T}, x \in (0, 1).$$

The last inequality implies that the differential equation is uniformly parabolic for $t \in [\hat{T}, \hat{T} + 2]$ (see [10]). Hence, by the standard theory we know that $u \in C^{2,1}((\hat{T}, \hat{T} + 2) \times [0, 1])$, and in particular $u_t(\hat{T} + 1, x) \in C([0, 1])$. Now we take a small $h > 0$. We have

$$\left\| u(\hat{T} + 1 + h, x) - u(\hat{T} + 1, x) \right\|_{L^\infty} \leq Ch$$

Due to the translation invariance of (1.1), for $t > \hat{T} + 1$, $u(t + h, x)$ is still a solution of (1.1). Applying a comparison principle, we obtain that

$$\|u(t + h, x) - u(t, x)\|_{L^\infty} \leq Ch.$$

Since h is arbitrary, we conclude that u_t remains bounded (also from below).

$$|u_t| \leq \left\| u_t(\hat{T} + 1) \right\|_{L^\infty} \quad \text{in } [\hat{T} + 1, \infty) \times [0, 1].$$

For the case $M = 0$, it is difficult to get a lower estimate on u_t without stronger assumption on the regularity of the initial data u_0 . So we shall assume that $u_0 \in C^2([0, 1])$. Let $A = (p - 1) \| |u_0|^{p-2} u_{0xx} \|_\infty + \| (u_0)_x^q \|_\infty$, then $At + u_0(x)$ is a supersolution of (1.1) and $u_0(x) - At$ is a subsolution of (1.1). Applying the comparison principle, we have $\|u(t, x) - u_0(x)\|_\infty \leq At$ and consequently for $t = h$ it follows that $\|u(h, x) - u_0(x)\|_\infty \leq Ah$. Combining the translation invariance of (1.1) and the comparison principle we get

$$\|u(t + h, x) - u(t, x)\|_{L^\infty} \leq \|u(h, x) - u_0(x)\|_\infty \leq Ah.$$

The boundedness of u_t follows immediately. In both cases we have

$$|u_t| \leq C_3, \quad \text{a.e. on } (\hat{T} + 1, +\infty) \times (0, 1). \quad (4.4)$$

Thanks to (4.1), we shall derive the following lemma providing a lower bound on the blow up profile of u_x in case of $u_x(t, 0)$ or $u_x(t, 1)$ becomes unbounded.

Lemma 4.2. *Let u be a global unbounded weak solution of (1.1). There exist $\nu, \eta > 0$ and $C_4 = C_4(C_3, p, q), C_5 = C_5(p, q) > 0$ with the following property. For all $(t, y) \in [t_1, +\infty) \times (0, 1/2)$ such that $u_x(t, y) \geq \nu > 0$, we have, for $y \leq x \leq y + \eta$*

$$\left[|u_x|^{p-2}u_x(t, x) + C_4 \right]^{\frac{p-1-q}{p-1}} \leq \left[|u_x|^{p-2}u_x^+(t, y) + C_4 \right]^{\frac{p-1-q}{p-1}} + C_5(x - y), \quad (4.5)$$

and for all $(t, y) \in [t_1, +\infty) \times (0, 1/2)$ such that $u_x(t, 1 - y) \leq -\nu < 0$, we have, for $y \leq x \leq y + \eta$

$$\left[-|u_x|^{p-2}u_x(t, 1 - x) + C_4 \right]^{\frac{p-1-q}{p-1}} \leq \left[-|u_x|^{p-2}u_x(t, 1 - y) + C_4 \right]^{\frac{p-1-q}{p-1}} + C_5(x - y). \quad (4.6)$$

Proof. Let $\nu > 0$ to be chosen later on and fix $(t, y) \in [t_1, +\infty) \times (0, 1/2)$ such that $u_x(t, y) \geq \nu$. Using the local continuity of u_x , we can define

$$\bar{x} := \max \left\{ x \in (y, 1] \mid u_x(t, x) \geq \frac{\nu}{2} \text{ on } [y, x] \right\}.$$

Note that $\bar{x} \in (y, 1]$ and that $u_x > \frac{\nu}{2}$ on $[y, \bar{x}]$. It follows from [10] that u_x is smooth on $[y, \bar{x}]$. Set $z(x) = |u_x|^{p-2}u_x(t, x) + C_3^{\frac{p-1}{q}}$, where C_3 is given by the estimate of $|u_t|$. Using the inequality (4.1), we get that, on $[y, \bar{x}]$ the function z satisfies in the classical sense

$$\begin{aligned} z' + z^{q/(p-1)} &= (|u_x|^{p-2}u_x(t, x))_x + \left(|u_x|^{p-2}u_x(t, x) + C_3^{\frac{p-1}{q}} \right)^{\frac{q}{p-1}} \\ &\geq (|u_x|^{p-2}u_x(t, x))_x + |u_x|^q + C_3 \\ &\geq 0. \end{aligned}$$

An integration yields $z(x)^{(p-1-q)/(p-1)} \leq z(y)^{(p-1-q)/(p-1)} + \left(\frac{q-p+1}{p-1} \right) (x-y)$, that is (4.5) with $C_4 = C_3^{\frac{p-1}{q}}$ and $C_5 = \frac{q-p+1}{p-1}$. It follows that

$$\left[|u_x|^{p-2}u_x(t, \bar{x}) + C_4 \right]^{\frac{p-1-q}{p-1}} \leq \left[|u_x|^{p-2}u_x(t, y) + C_4 \right]^{\frac{p-1-q}{p-1}} + C_5(\bar{x} - y).$$

Now if $\bar{x} < 1$, then $u_x(t, \bar{x}) = \frac{\nu}{2}$. Using that $u_x(t, y) \geq \nu$, we get

$$\left[\left(\frac{\nu}{2} \right)^{p-1} + C_4 \right]^{\frac{p-1-q}{p-1}} - [\nu^{p-1} + C_4]^{\frac{p-1-q}{p-1}} \leq C_5(\bar{x} - y).$$

Taking a suitable ν , we can find a small $\eta > 0$ (with both ν and η depending only on C_3, p, q) such that

$$\left[\left(\frac{\nu}{2} \right)^{p-1} + C_4 \right]^{\frac{p-1-q}{p-1}} - [\nu^{p-1} + C_4]^{\frac{p-1-q}{p-1}} \geq C_5\eta.$$

It follows that $\bar{x} - y \geq \eta$, leading to the desired result (the case $\bar{x} = 1$ being obvious). The estimate (4.6) follows similarly by considering $z(x) = -|u_x|^{p-2}u_x(t, 1-x) + C_3^{\frac{p-1}{q}}$.

Remark 4.1. For $M > 0$ we have seen that the global solution became classical for large time and Lemma 4.2 could be proved easily. For $M = 0$, since the solution is only a weak solution, we needed to give a sense to the equation satisfied by $z(x)$ making the proof of Lemma 4.2 a little bit more technical.

Proof of theorem 1.1 for $0 \leq M < M_b$.

Assume that u is a global weak solution which is unbounded in $W^{1,\infty}$. We know that when $t \rightarrow \infty$, u converges to $W = V_M$ in $C[0, 1]$ and in $C^1[\delta, 1 - \delta]$ for all $\delta > 0$. Since u_x is unbounded and can only blow up either at $x = 0$ or at $x = 1$, there exist sequences $t_n \rightarrow \infty$, $x_n \rightarrow 0$ and/or $z_n \rightarrow 1$ such that

$$u_x(t_n, x_n) \rightarrow +\infty \quad \text{and/or} \quad u_x(t_n, z_n) \rightarrow -\infty \quad (4.7)$$

We consider only the first case, the other one being similar. Taking $t = t_n$ and $y = x_n$ in (4.5)-(4.6) and sending $n \rightarrow \infty$, we deduce that, for any $x \in (0, \eta)$

$$[|W_x(x)|^{p-2}(W_x)^+ + C_4]^{\frac{p-1-q}{p-1}} \leq C_5 x.$$

This would imply that

$$|W_x|^{p-2}(W_x)_+ + C_4 \geq (C_5 x)^{\frac{1-p}{q-p+1}}.$$

Passing to the limit $x \rightarrow 0$ we get a contradiction since $W = V_M \in C^1([0, 1])$. So all the global solution are bounded in $W^{1,\infty}$.

Now for $t > t_1$, we have $\|(|u_x|^{p-2}u_x)_x\|_{L^\infty} = \|u_t - |u_x|^q\|_{L^\infty} \leq C$. It follows that the family $\{|u_x|^{p-2}u_x(t, \cdot)\}_{t \geq t_1}$ is pre-compact in $C([0, 1])$, thus $|u_x|^{p-2}u_x(t, \cdot)$ converges to $|W_x|^{p-2}W_x$ when $t \rightarrow \infty$. If $M > 0$, using that $W_x \geq 2\eta > 0$, we conclude that u_x converges to W_x uniformly on $[0, 1]$. If $M = 0$, we conclude that u_x converges to 0 uniformly on $[0, 1]$.

Proof of theorem 1.1 for $M > M_c$.

This is an immediate consequence of Proposition 3.3 and the fact that (2.5) admits no solution for $M > M_b$.

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